THE STABILITY OF A SWARM OF METEORITES AND OF A PLANET AND SATELLITE.

THE problem of the stability of a swarm of meteorites which is under the action of its own gravity and the attraction of the sun, that is to say the determination of the condition under which the swarm will remain unbroken up by the tidal action of the sun, has been dealt with by Schiaparelli, M. Luc Picard and M. Charlier (*Bulletin de l'Académie de St. Petersbourg*, t. xxxii. No. 3). The result obtained by the first assigns a much wider limit of stability to such a system than that arrived at by the other two investigators mentioned; but there cannot, I think, be any doubt of the greater correctness in actual cases of the narrower limit. The problem is intimately related to the still more interesting question of the stability of the earth-moon system, which was treated by Mr. G. W. Hill in a remarkable paper in the *American Journal of Mathematics* for 1878. This again is, in another form, the problem treated still earlier with a special object in view by Edouard Roche (*Mém. Acad. de Montpellier*, vol. i. 1847-50; see also *Annales de l'Observatoire*, t. v.), when he arrived at his result concerning the limiting relation between the distance of a satellite from a primary and the diameter of the primary, which must hold in order that the satellite, held together by its own gravitation only, may just not break up under the tidal forces due to the primary, and his corresponding result for a planet's or satellite's atmosphere.

These investigations, though of great general interest, are not so well known as might be expected, and one object of this paper is to give some slight account of them. An abstract of the work of Charlier and Hill is given also in Dr. Routh's recently published work on "Dynamics of a Particle." But I wish also to point out how the main conclusions of Charlier, that of Roche with respect to a planet's atmosphere, and more indirectly the result of Hill, can be obtained by means of elementary considerations.

The problems just referred to have been treated by very different methods. Schiaparelli's discussion is a direct attack of a somewhat long and involved nature; those of MM. Picard and Charlier (*Bulletin de l'Académie de St. Petersbourg*, t. xxxii. No. 3; see also Tisserand, *Mécanique Céleste*, t. iv.) make use of the method of revolving axes. The radius vector from the centre of the sun to the centre of the meteoric swarm is supposed to revolve with angular velocity, n say, about the centre of the sun as a fixed point; then the motion of a particle of the swarm is referred to three directions at right angles to one another having their origin at the centre of the swarm, and turning with the radius vector just specified. These axes may be taken as an axis of ξ towards the sun, an axis of η at right angles to this in the plane of motion of the centre, and an axis of ζ at right angles to this plane. Then equations of motion relative to these moving axes are written down for a particle the component distances of which from the centre are ξ, η, ζ, it being supposed in the first place that the distance r of the centre of the swarm from the sun, and the angular velocity n of the radius vector are both variable. Approximate values of the forces are obtained by supposing that ξ, η, ζ are small in comparison with r, and that r, and therefore also n, is constant. When account is taken of the condition that must hold for the central particle, the equations assume the very simple form

$$\ddot{\xi} - 2n\dot{\eta} - (3n^2 - \mu)\xi = 0, \quad \ddot{\eta} + 2n\dot{\xi} + \mu\eta = 0, \quad \ddot{\zeta} + (n^2 + \mu)\zeta = 0.$$

The value of μ is $\frac{4}{3}\pi ks$, where k is the gravitation constant and s is the average density of the portion of the swarm within the spherical surface on which the particle lies, supposed symmetrical about the centre. Considering only particles in the plane of ξ, η, the values of these co-ordinates are supposed to oscillate about certain constant values, so that $\xi = a \cos(\omega t + \epsilon)$, $\eta = b \sin(\omega t + \epsilon)$. That is, each particle is supposed to revolve in an ellipse, the centre of which is the centre of the swarm, and of which one axis is along the line of centres and the other perpendicular to that line. The ellipse is a circle if $a = b$, and ω is then the angular velocity of the *relative* motion of the particle about the centre. These values substituted in the first two equations of motion lead to the condition

$$(\omega^2 - \mu)(\omega^2 + 3n^2 - \mu) - 4\omega^2 n^2 = 0.$$

Now ω is $2\pi f$, if f be the frequency of oscillation; and if the oscillation be stable, f will have a positive real value. The roots of the quadratic in ω^2 just written must therefore be real and positive; and it is not hard to see that the required condition for this is $\mu > 3n^2$. This gives, when μ is replaced by $\frac{4}{3}\pi ks$, and n^2 by kM/r^3, where M is the mass of the sun (for we have for the central particle $n^2 r = kM/r^2$), the inequality

$$\tfrac{4}{3}\pi s r^3 > 3M.$$

In order, therefore, that the swarm of small particles may keep together, it is necessary that its average density be greater than that of a spherical distribution of matter of radius equal to the sun's distance and of three times the sun's mass. The problem for an elliptic orbit of eccentricity e has been considered by M. Callandreau (*Bulletin Astronomique*, 1896). The condition $\mu > 3n^2$ is in this case replaced by $\mu > 3n^2 + 5e^2n^2$. The swarm is therefore rendered less stable by the eccentricity.

It is to be remembered that the effect of the distortion of the swarm by the tidal force of the sun is neglected, and it does not seem of much importance to consider eccentricity of orbit so long as the assumption of sphericity of figure is maintained.

Since the equation of condition stated above must be satisfied, only values of ω consistent therewith, and with the inequality $\mu > 3n^2$, are admissible. Thus if $\omega = 0$, that is if there be no revolution of any particle about the centre of the swarm, the equation gives $\mu = 3n^2$, and the inequality is not fulfilled. This is a limiting case between stability and instability.

Now let the differential equations of motion referred to above (from which the more roughly approximate equations quoted are derived) be modified for the case in which the swarm is replaced by a planet of given mass m, and the particle considered by a satellite of mass m' at the external point ξ, η, ζ, then let them be multiplied by $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ respectively, integrated, and added. Thereby will be obtained the equation of kinetic energy for the relative motion, commonly called Jacobi's equation. This has, if $\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2$, the square of the resultant relative velocity, be denoted by v^2, the form

$$v^2 - \left[2\frac{\mu}{\rho} + 2\frac{r^2n^2}{r_1} + n^2\left\{(r-\xi)^2 + \eta^2\right\}\right] + C = 0$$

where

$$\mu = k(m + m'),\ \rho = \sqrt{\xi^2 + \eta^2 + \zeta^2},\ r_1 = \sqrt{(r-\xi)^2 + \eta^2 + \zeta^2},$$

C is a constant, and r is, as before, the distance of the centre of the sun from that of the planet.

Now when v^2 has a given value, the satellite must have its centre on the surface of which the equation is obtained by placing that value in the equation just written. Hence, since v^2 is positive, the satellite cannot pass across the surface for which $v^2 = 0$, that is the surface for which

$$2\frac{\mu}{\rho} + 2\frac{r^3n^2}{r_1} + n^2\left\{(r-\xi)^2 + \eta^2\right\} - C = 0;$$

by putting $\zeta = 0$ in this we obtain the equation of the curve in which the surface intersects the plane of ξ, η. An investigation of the surface shows that if C be positive the surface consists of three sheets, of which two are closed and surround the sun and the planet respectively; and the third is asymptotic to a surface of revolution about an axis passing through the sun's centre perpendicular to the ecliptic, and surrounds the two closed surfaces. Within the closed surfaces, or outside the third surface, v^2 is positive; between the closed surfaces and the outer asymptotically cylindrical surface v^2 is negative, and therefore v is imaginary. The satellite must therefore be within one of the closed surfaces, or beyond the outer surface; in either case it cannot cross the surface of zero velocity.

When the proper values of the quantities for the earth-moon system are inserted, it is found that the moon is within the closed sheet surrounding the earth, from which, therefore, it cannot escape. The distance of the moon's centre from the earth, Mr. Hill has calculated, cannot exceed 109·694 equatorial radii of the earth. The result is based, of course, on the assumption that the eccentricity of the earth's orbit may be neglected.

If, besides neglecting the eccentricity, we suppose the moon to move in the plane of the ecliptic, and to be so distant that we may neglect terms in η, the equation of the curve of no velocity in the plane of the ecliptic is

$$\frac{\mu}{\rho} + \frac{3}{2}n^2\xi^2 = c,$$

or if $\xi = \rho\cos\theta$

$$3n^2\cos^2\theta . \rho^3 - c\rho + 2\mu = 0,$$

where c is another constant.

The roots of this cubic in ρ are all real if $\cos^2\theta > c^3/81n^2\mu^2$, and the rule of signs shows that there is only one negative root. The curve of no velocity consists then of a closed branch round the origin of co-ordinates, the centre, E, of the earth in the present case. Besides this there are two infinite branches which are asymptotic to the parallel lines AB, A'B' represented by

$$3n^2\xi = c.$$

Thus the curve is as roughly represented in Fig. 1. The line CD shows the direction of the radius vector from the sun.

Between the closed curve and the infinite branches v is imaginary, and the satellite must be either within the closed branch or beyond the boundary represented by the infinite branches. The calculation gives very approximately 110 equatorial radii of the earth for the greatest distance of any point of the closed branch from the centre. The form of this branch is that of an oval, being slightly longer in the direction towards the sun than in the transverse direction.

The theorem of Roche which we discuss here is contained in the statement that the atmosphere of a satellite cannot be held together merely by the gravitational attraction of the satellite unless the inequality

$$\frac{m}{M} > (2 + c)\frac{a^3}{r^3}$$

is fulfilled, in which m is now the mass of the satellite, M that of the planet, c the ratio of the square of the angular velocity

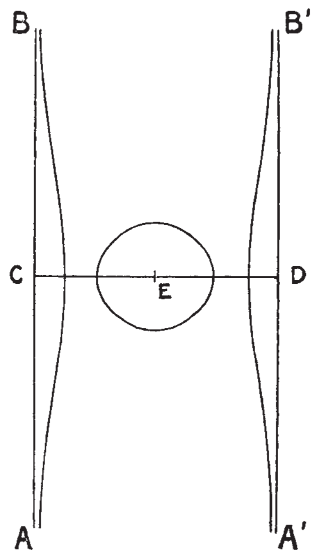

FIG. 1.

of the satellite's axial rotation to the square of the angular velocity of its orbital revolution, a denotes the satellite's radius, and r the distance of the centre of the satellite from the planet. If the densities of the planet and satellite be s_1, s, and their radii a_1, a, and c be unity, that is if the satellite turns always the same face to the primary, we have for the inequality

$$\frac{r^3}{a_1^3}\frac{s}{s_1} > 3, \text{ or } r > 1{\cdot}44 a_1 \sqrt[3]{s_1/s}.$$

It should be mentioned that Roche's investigations embraced much more than this; they included the determination of the figure of a fluid satellite, and entered into other matters which cannot be discussed here.

To deal with these questions in an elementary way, consider the important particular case of a spherical swarm of radius a moving round the sun, and turning as a whole about an axis perpendicular to the orbit in the period of revolution, so that it turns the same face always towards the sun. This is, of course, a less general problem than that considered above; it is indeed the case of that problem in which ω is zero It is interesting to see from the more general investigation that the condition obtained by the consideration of this case is sufficient to give stability for any value of ω provided it fulfils the equation of condition stated above. We shall obtain also by the elementary process a wider condition for the case in which ω is not zero. This will give the inferior limit assigned by Roche to the distance of a satellite from its primary.

A particle, of unit mass, say, at the centre, C (Fig. 2), at distance SC ($=r$) from the sun, is in relative equilibrium under the sun's attraction and the so-called centrifugal force. That is, we have for that particle

$$\frac{kM}{r^2} - n^2 r = 0.$$

Again, a particle on the outside of the swarm at the point nearest the sun is at a distance $r - a$, and under attraction $kM/(r-a)^2$. Hence there is a preponderance of attraction over the acceleration $n^2(r-a)$ towards S. This excess is

$$\frac{kM}{(r-a)^2} - n^2(r-a) = kM\left\{\frac{1}{(r-a)^2} - \frac{1}{r^2} + \frac{a}{r^3}\right\}$$
$$= 3kM\frac{a}{r^3}$$

nearly. This must at least be balanced by the attraction towards the centre, C, exerted by the swarm, if the particle is not to leave the swarm. Hence we must have $\frac{4}{3}\pi ska^3/a^2 > 3kMa/r^3$, or

$$\tfrac{4}{3}\pi s r^3 > 3M,$$

as before. The same result would be obtained for a particle at B. In that case the attraction of the sun $kM/(r+a)^2$ would be insufficient to supply the acceleration $n^2(r+a)$ towards the sun. The condition that this should be supplied by the attraction of the swarm is that $\frac{4}{3}\pi s r^3$ should be at least equal to 3M.

This result holds, of course, for all particles within the swarm on the line SC, for no particle experiences any force on the whole from the spherical layer outside it.

It is to be observed that a particle at A or B (or on the line SC) is in greater danger of leaving the swarm from the causes

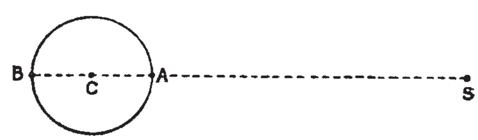

FIG. 2.

just explained, than a particle elsewhere on the spherical surface.

If the particles of the swarm have other angular velocities than that supposed above about an axis through its centre perpendicular to the plane of the orbit, the investigation will run as follows. Suppose applied to each of the particles a force per unit mass equal and opposite to that, $n^2 r$, exerted by the sun on the central particle. This will have no effect on the relative motions of the particles or on the figure of the swarm. Upon the particle nearest the sun the force per unit mass toward the sun is now

$$\frac{kM}{(r-a)^2} + \omega_1^2 a - n^2 r = 2kM\frac{a}{r^3} + \omega_1^2 a$$

if ω_1 be the angular velocity *in space* of the radius vector drawn from the centre to the particle (that is, not the relative angular velocity ω above, but $\omega + n$). This must at least be balanced by the attraction towards the centre exerted by the swarm if the particle is not to leave it. Thus we have $\frac{4}{3}\pi ksa > 2kMa/r^3 + \omega_1^2 a$, or since $kM = n^2 r^3$

$$\tfrac{4}{3}\pi s r^3 > \left(2 + \frac{\omega_1^2}{n^2}\right)M.$$

Thus if the swarm as a whole make one rotation in the period of revolution round the sun, $\omega_1^2/n^2 = 1$, and we obtain the same result as before.

Let now the swarm of particles be replaced by a spherical planet with an atmosphere composed of discrete small particles, the whole being held together by gravitational attraction alone. Then if the mass of the planet be denoted by m, the inequality $\frac{4}{3}\pi s r^3 > 3M$ becomes $\frac{4}{3}\pi s a^3 > 3Ma^3/r^3$, that is $m/M > 3a^3/r^3$. This is to be fulfilled if the atmosphere is not to be dissipated by tidal action. The same thing will, of course, hold for a primary and the atmosphere of a satellite.

In the more general case, that in which the satellite has rotational velocity ω about its axis (supposed perpendicular to the plane of the orbit), we have, assuming that the satellite is spherical and denoting ω_1^2/n^2 by c, $m/M > (2+c)a^3/r^3$. This agrees with the former result when $c=1$. These results were first obtained by M. Roche.[1]

The figure of a fluid satellite is determined by finding a surface to which the resultant of the gravitational pull of the primary on unit mass, a force n^2r equal and opposite to the gravitational pull on unit mass at the centre, the gravitational force per unit mass exerted by the matter of the satellite itself, and the centrifugal force of unit mass, is everywhere perpendicular. A first approximation to the force due to the satellite itself is obtained by neglecting the deviation from sphericity, as is done above. But into this discussion we cannot here enter. It can only be stated that the final result, taking into account the distortion of the satellite, is that the satellite will be broken up if it approaches closer to the primary than the limit given by the inequality

$$r > 2{\cdot}44 \sqrt[3]{s_1/s_1}.$$

Now imagine a planet and a satellite moving round the sun, the satellite being destitute of relative velocity. The satellite may, for example, be regarded as a particle (of unit mass say) of a ring of small mass composed of particles surrounding the primary at a distance a, the whole turning, if that were possible, with angular velocity equal to that of the primary round the sun. By what we have seen above, the excess of solar attraction over the sunward acceleration is, for a particle on the side nearest the sun, $3kMa/r^3$. This must be balanced by the attraction km/a^2, so that we have the equality $km/a^2 = 3kMa/r^3$.

From these expressions for the forces we see that the potential energy, with a term for centrifugal force included, may be taken, for unit mass of the infinitesimal satellite on the line of centres at the point nearest the sun, as $-km/a - \frac{3}{2}kMa^2/r^3$. This is an example of the almost self-evident principle, known as the theorem of Coriolis, that if there be included a term in the potential energy which will give the components of centrifugal force, we may write down the equation of relative kinetic energy, just as if the rotating axes were fixed. The potential energy thus required for the centrifugal force on a particle of unit mass at a point at distances ξ, η, ζ from the planet's centre, measured respectively along the line of centres, perpendicular to this line in the plane of motion, and perpendicular to the plane of motion, is $-\frac{1}{2}n^2\{(r-\xi)^2+\eta^2\}$, or, since $n^2r = kM/r^2$,

$$-\tfrac{1}{2}kM\{(r-\xi)^2+\eta^2\}/r^3.$$

The total potential energy being taken as

$$-\left[\frac{km}{\rho}+\frac{kM}{r_1}+\frac{1}{2}\frac{kM}{r^3}\left\{\left(r-\xi\right)^2+\eta^2\right\}\right]$$

(with, as before,

$$v^2 = \dot\xi^2+\dot\eta^2+\dot\zeta^2,\ \rho^2=\xi^2+\eta^2+\zeta^2,\ r_1^2=(r-\xi)^2+\eta^2+\zeta^2,$$

and $n^2 = kM/r^3$), the equation of relative kinetic energy is

$$\frac{1}{2}v^2-\left[\frac{km}{\rho}+\frac{kM}{r_1}+\frac{1}{2}\frac{kM}{r^3}\left\{\left(r-\xi\right)^2+\eta^2+\zeta^2\right\}\right]+C=0,$$

which, for an infinitesimal satellite, is Hill's equation as given above.

For the moon, which has mass m' sensible in comparison with the mass, m, of the earth, the first term in the square brackets should be $k(m+m')/\rho$.

It may be noticed by the dynamical student that if the above expression for the potential energy be denoted by V, we have not $\ddot\xi = -\partial V/\partial\xi$, &c., for the equations of motion, but

$$\ddot\xi - n\dot\eta = -\partial V/\partial\xi,\ \ddot\eta + n\dot\xi = -\partial V/\partial\eta\ \ \ddot\zeta = -\partial V/\partial\zeta.$$

A. GRAY.

[1] I have found since the above was written that the same elementary view of this matter is given by Roche himself in his paper "Recherches sur les Atmosphères des Comètes," *Annales de l'Observatoire*, t. v. 1859. Perhaps I may here direct attention to a valuable paper by Roche (which may, however, though I have not seen it referred to, be well known to astronomers), entitled "Essai sur la Constitution du Systeme Solaire," *Mém. de l'Acad. de Montpellier*, t. viii. This gives a general account of the author's cosmogonic researches.

ANTELOPES AND THEIR RECOGNITION MARKS.

THE Tragelaphine Antelopes hold a unique position amongst the hollow-horned ruminants. No other group can show species so sharply contrasted in size and build as the massive eland rising over sixty inches at the withers. and the dainty little bush-buck which falls short of half that height. Only the Indian black buck amongst the gazelles can match the nylghaie and nyala for diversity of sexual colouring; and for elegance of form, coupled with beauty of marking and grandeur of carriage, the kudu is surpassed by no species of mammal.

Apart from certain features presented by the skull and horns, the affinity between the species here mentioned is attested by the markings of the skin. On a ground-colour shading from slate to chestnut are distributed certain white spots, patches or stripes, which crop up so persistently in the different genera as to leave no doubt they are a heritage from a common ancestor. A comparison between the skins of the existing species suggests that this ancestor was coloured somewhat as follows:—Body and head yellowish red; flanks and hind-quarters striped with white; on the throat two white patches, one at each end; one or two spots on the cheeks, a V-shaped stripe between the eyes, a white chin, a white upper lip; legs paler on the inner side, quite white at base close to chest and groin, and with two white spots on the pasterns in front.

Some or all of these markings have been inherited with scarcely an exception by every known species of Tragelaphine. Sometimes the spots on the head, sometimes the stripes on the body, sometimes the patches on the throat are suppressed; but even in extreme cases of suppression, a spot here, a stripe there, persists as a tell-tale sign of descent. The usefulness of characters so constant may be taken for granted. The nature of their usefulness has been discussed by both Wallace and Darwin; but so great is the discord between the opinions of these authorities that one cannot think both are right.

Referring to the importance of special marks for recognition where many species of nearly the same size and general form inhabit the same region, Mr. Wallace says: "It is interesting to note that these markings for recognition are very slightly developed in the antelopes of the woods and marshes. . . . The wood-haunting bosch-bok (*T. sylvaticus*) goes in pairs, and has hardly any distinctive markings on its dusky chestnut coat, but the male alone is horned. The large and handsome kudu frequents brushwood, and its vertical white stripes are no doubt protective, while its magnificent spiral horns afford easy recognition. The eland, which is an inhabitant of the open plain, is uniformly coloured, being sufficiently recognisable by its large size and distinctive form; but the Derbyan eland is a forest animal, and has a protectively striped coat. In like manner, the fine Speke's antelope, which lives entirely in the swamps and among reeds, has pale vertical stripes on the sides (protective), with white markings on face and breast for recognition" ("Darwinism," p. 220).

It may be inferred from this passage that the interest attached to the slight development of recognition marks in the antelopes of the woods and marshes lies in the needlessness of such marks for species living apart and not herding with others of the same general size and form. If, however, there is no likelihood of confusion, it is not quite clear from what species the horns of the kudu serve to distinguish their owner, nor what significance in this connection is to be attributed to the occurrence of horns only in the male of the bosch-bok. Similarly, it is not clear what use Speke's marsh-buck can have for recognition marks. If, however, the spots on the face and throat subserve recognition in this species, we must also conclude they are retained for that purpose in the bongo (*T. euryceros*), the lesser kudu, the nyala (*T. angasi*), in which they are very conspicuous, as well as in the various smaller kinds of bush-buck, which in other parts of Africa live the same life as the bosch-bok of the Colony. Surely, too, Derby's eland is at least as recognisable by its large size and distinctive form as the Cape species; yet it is adorned with a conspicuous V-shaped stripe between the eyes, and the lower throat-patch forms a white collar, standing boldly out against the black hue of the neck.

In short, if the marks in question have been preserved for recognition, it is singular that they are exceptionally well developed in the species that live in pairs or small parties by themselves in thick bush—species which, according to the hypothesis, have little, if any, need of them. It is conceded, of course, that the spots on the head and throat, like the stripes on the body,